Journal of Applied Mechanics and Technical Physics, Vol. 44, No. 1, pp. 141–145, 2003

CONSTITUTIVE RELATIONS OF NONLINEAR THERMOELASTICITY OF ANISOTROPIC BODIES

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UDC 539.3

Quasilinear relations for finite reversible deformation of anisotropic materials are obtained using a thermomechanical approach. Free energy is written as a function of temperature and compatible invariants of the logarithmic strain measure and basis tensors. Nonlinear thermomechanical effects including different types of material behavior in tension and compression and the temperature dependence of the elastic tensor are taken into account.

Key words*: nonlinear thermoelasticity, anisotropy, constitutive relations, finite strain of a macrovolume.*

The basic concepts of anisotropic thermoelasticity for infinitesimal strains are formulated in [1]. Some problems of nonlinear elasticity of anisotropic materials are considered in [2]. In the present paper, a variant of constitutive relations between stresses, finite strains, and temperature in an anisotropic elastic material are obtained using the general thermodynamic approach proposed by Il'yushin [3] and Sedov [4] for studying deformation processes.

We consider a representative macrovolume dV_0 of density ρ_0 at a temperature T_0 in an anisotropic body in the initial state. Heat input $d'Q$ and displacements of the macrovolume walls, determined by the strain affinor $\Phi(t)$, produce the temperature $T(t)$ and stress $S(t)$ fields in the macrovolume. The strain, stress, and temperature fields inside the macrovolume are assumed to be homogeneous.

We consider homogeneous finite strain of the macrovolume. Transition of an elastic material from the initial state at $t = t_0$ to an arbitrary final state at $t = t_k$ is determined by time variation of the left distortion measure $U(t)$ [5]. We confine our attention to deformation processes for which $U(t) = U_i(t)\boldsymbol{a}_i^{(k)}\boldsymbol{a}_i^{(k)}$, where $\boldsymbol{a}_i^{(k)}$ are the unit vectors of the principal strain axes at the time $t = t_k$ and $U_i(t)$ are the principal elongations. For these processes, the principal strain axes coincide with the same material fibers. For convenience, we use the Hencky tensor $H = \ln U$ [5] as a strain measure, which allows us to consider the volume-change and distortion processes separately. The volume change is characterized by the first invariant of the Hencky measure $\ln(dV/dV_0) = \theta = H : E$, whereas the distortion is described by the deviator of this measure $\tilde{H} = H - (1/3)\theta E$ (E is the unit tensor). As a stress measure, we use the generalized tensor of true stresses $\Sigma = (dV/dV_0)S$, which is energy conjugate with the Hencky measure. The first invariant of this tensor determines the hydrostatic stress in the macrovolume $-p = \Sigma : E$.

We associate the deformation process of an anisotropic material with its image in Il'yushin's six-dimensional space [3]. In this space, the tensors H and Σ are represented by the six-dimensional vectors $\boldsymbol{h}$ and $\boldsymbol{\sigma}$, respectively. The vectors $\boldsymbol{h}$ and $\boldsymbol{\sigma}$ are determined by the relations of [3] from the components of the tensors H and Σ in the coordinate system whose axes coincide with the principal axes of the initial material anisotropy with unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. The basis vectors $\boldsymbol{i}_0$, $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, $\boldsymbol{i}_3$, $\boldsymbol{i}_4$, and $\boldsymbol{i}_5$ of Il'yushin's space are images of the tensors of the canonical basis:

$$\begin{gathered} I_0 = (\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_3)/\sqrt{3}, \quad I_1 = (2\boldsymbol{e}_3\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_1 - \boldsymbol{e}_2\boldsymbol{e}_2)/\sqrt{6}, \\ I_2 = (\boldsymbol{e}_1\boldsymbol{e}_1 - \boldsymbol{e}_2\boldsymbol{e}_2)/\sqrt{2}, \quad I_3 = (\boldsymbol{e}_1\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_1)/\sqrt{2}, \\ I_4 = (\boldsymbol{e}_2\boldsymbol{e}_3 + \boldsymbol{e}_3\boldsymbol{e}_2)/\sqrt{2}, \quad I_5 = (\boldsymbol{e}_1\boldsymbol{e}_3 + \boldsymbol{e}_3\boldsymbol{e}_1)/\sqrt{2}. \end{gathered} \tag{1}$$

Tula State University, Tula 300600. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 44, No. 1, pp. 170–175, January–February, 2003. Original article submitted March 6, 2002; revision submitted June 10, 2002.

In the coordinate system associated with the principal axes of the material anisotropy, the fourth-rank tensor N^{IV}, which determines the properties of the anisotropic medium, has a canonical form. The image of the tensor N^{IV} is a second-rank tensor n; the components of these tensors are related by the direct and inverse relations [6] $(3/2)N_{ijkl} = \beta^{\alpha}_{ij} n_{\alpha\beta} \beta^{kl}_{\beta}$ and $(3/2)n_{\alpha\beta} = \beta^{ij}_{\alpha} N_{ijkl} \beta^{\beta}_{kl}$ (where β^{ij}_{α} and β^{α}_{ij} are the known transition matrices [3]).

For anisotropic materials with different types of symmetry properties, one can determine invariant subspaces whose basis vectors remain unchanged under orthogonal transformations. The bases of these invariant subspaces form the following vector sets:

1) $\boldsymbol{i}_0$, $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, $\boldsymbol{i}_3$, $\boldsymbol{i}_4$, and $\boldsymbol{i}_5$ for triclinic material;
2) $\boldsymbol{i}_0$, $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, and $\boldsymbol{i}_3$ for monoclinic material;
3) $\boldsymbol{i}_0$, $\boldsymbol{i}_1$, and $\boldsymbol{i}_2$ for rhombic (orthotropic) material;
4) $\boldsymbol{i}_0$ and $\boldsymbol{i}_1$ for tetragonal, trigonal, and hexagonal (transversely isotropic) materials;
5) $\boldsymbol{i}_0$ for Cubic and isotropic materials.

The existence of these sets of invariant basis vectors is implied by the invariance of the tensors of the canonical basis (1) under orthogonal transformations of physical space.

For reversible processes, the basic thermomechanical relation of [3] has the form

$$\dot{\psi} + \eta \dot{T} = (1/\rho_0)\boldsymbol{\sigma} \cdot \dot{\boldsymbol{h}}, \tag{2}$$

where ψ and η are the free energy and entropy per unit mass, respectively, T is the absolute temperature, and $(1/\rho_0)\boldsymbol{\sigma} \cdot \dot{\boldsymbol{h}}$ is the specific stress power.

Let the specific free energy be a function of strain and temperature: $\psi = \psi(\boldsymbol{h}, T)$. Then,

$$\dot{\psi} = \frac{\partial \psi}{\partial \boldsymbol{h}} \cdot \dot{\boldsymbol{h}} + \frac{\partial \psi}{\partial T} \dot{T}. \tag{3}$$

Comparing relations (2) and (3), we obtain the expressions for stresses and entropy

$$\frac{1}{\rho_0} \boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{h}}, \qquad \eta = -\frac{\partial \psi}{\partial T}, \tag{4}$$

which are implied by the thermomechanical approach [3, 4].

We write the free energy in the form accepted in linear anisotropic thermoelasticity [1]:

$$\psi = (1/2\rho_0)\boldsymbol{h} \cdot n \cdot \boldsymbol{h} - (1/\rho_0)\boldsymbol{b} \cdot \boldsymbol{h}(T - T_0) + C_h (T - T_0)^2/(2T_0). \tag{5}$$

We assume that the tensor n depends on strains: $n = n(\boldsymbol{h})$ and the vector $\boldsymbol{b}$ depends on temperature: $\boldsymbol{b} = \boldsymbol{b}(T)$. For constant strains, the specific heat is denoted by C_h.

Differentiation of (5) with allowance for (4) yields

$$\boldsymbol{\sigma} = n(\boldsymbol{h}) \cdot \boldsymbol{h} + \frac{1}{2} \boldsymbol{h} \cdot \frac{dn}{d\boldsymbol{h}} \cdot \boldsymbol{h} - \boldsymbol{b}(T - T_0); \tag{6}$$

$$\eta = \frac{1}{\rho_0} \Big(\boldsymbol{b} + \frac{\partial \boldsymbol{b}}{\partial T}(T - T_0) \Big) \cdot \boldsymbol{h} - \frac{C_h}{T_0}(T - T_0).$$

To make the constitutive relations (6) more specific, we write the tensor function $n(\boldsymbol{h})$ in the form of dyadic expansion

$$n(\boldsymbol{h}) = c + \sum_{\alpha=0}^{m-1} c_\alpha (\boldsymbol{i}_\alpha \boldsymbol{h} + \boldsymbol{h} \boldsymbol{i}_\alpha). \tag{7}$$

Here c is the constant tensor of the material properties, c_α are the material constants, m is the dimension of the invariant subspace for the material, and $\boldsymbol{i}_\alpha$ are the basis vectors of invariant subspaces (see sets 1–5).

The dimension m of the invariant subspace for anisotropic materials of different types is determined by the number of basis vectors in sets 1–5. For example, $m = 1$ for an isotropic material, $m = 2$ for a transversely isotropic material, and $m = 3$ for an orthotropic material. Relations (7) may be used not only for these types of materials but also for media with more complex anisotropic properties, for example, monoclinic and triclinic media.

We find the derivative $dn/d\boldsymbol{h}$ that enters (6), assuming that the basis vectors $\boldsymbol{i}_\alpha$ remain unchanged during deformation:

$$\frac{dn}{d\boldsymbol{h}} = \sum_{\alpha=0}^{m-1} c_\alpha \Big(\boldsymbol{i}_\alpha \frac{d\boldsymbol{h}}{d\boldsymbol{h}} + \frac{d\boldsymbol{h}}{d\boldsymbol{h}} \boldsymbol{i}_\alpha\Big), \qquad \frac{d\boldsymbol{h}}{d\boldsymbol{h}} = \sum_{\alpha=0}^{5} \boldsymbol{i}_\alpha \boldsymbol{i}_\alpha = E_6$$

(E_6 is the unity tensor in the six-dimensional space).

Combining relations (6) and (7), we obtain the stress vector

$$\boldsymbol{\sigma} = \Big[c + \sum_{\alpha=0}^{m-1} c_\alpha(\boldsymbol{i}_\alpha \boldsymbol{h} + \boldsymbol{h}\boldsymbol{i}_\alpha + h_\alpha E_6)\Big] \cdot \boldsymbol{h} - \boldsymbol{b}(T - T_0), \tag{8}$$

where $h_\alpha = \boldsymbol{i}_\alpha \cdot \boldsymbol{h}$.

Setting the constants c_α in (8) equal to zero and assuming that the vector $\boldsymbol{b}$ is constant, we obtain generalization of the Duhamel–Neumann relations [1] to the case of finite strains.

It follows from relations (6) that the vector $\boldsymbol{b}(T - T_0) = \boldsymbol{\sigma}|_{\boldsymbol{h}=\boldsymbol{0}}$ is the vector of temperature stresses that occur in the macrovolume due to a temperature change for zero strains. We find a relation between the vector function $\boldsymbol{b}(T)$ and the temperature strains of the macrovolume $\boldsymbol{h}^T$. The temperature strains are understood as strains that occur in a nonconstrained macrovolume for $\boldsymbol{\sigma} = \boldsymbol{0}$ due to a temperature change:

$$\boldsymbol{h}^T = \boldsymbol{h}\Big|_{\boldsymbol{\sigma}=\boldsymbol{0}}.$$

Setting $\boldsymbol{\sigma} = \boldsymbol{0}$ in (8), we obtain

$$\boldsymbol{b}(T - T_0) = \Big[c + \sum_{\alpha=0}^{m-1} c_\alpha(\boldsymbol{i}_\alpha \boldsymbol{h}^T + \boldsymbol{h}^T \boldsymbol{i}_\alpha + h_\alpha^T E_6)\Big] \cdot \boldsymbol{h}^T, \tag{9}$$

where $h_\alpha^T = \boldsymbol{i}_\alpha \cdot \boldsymbol{h}^T$.

Relation (9) expresses the vector function $\boldsymbol{b}(T)$ that enters (8) in terms of temperature strains, which can be determined experimentally and represented in the form of a certain function of temperature. In particular, the vector $\boldsymbol{h}^T$ can be written in the form accepted in the linear theory

$$\boldsymbol{h}^T = \boldsymbol{a}(T - T_0), \tag{10}$$

where $\boldsymbol{a}$ is the direction vector of temperature strains.

The vector $\boldsymbol{a}$ in (10) is the image of the tensor of thermal-expansion coefficients A in Il'yushin's space. Substituting (10) into (9), we obtain the following relation between the direction vectors of temperature stresses and temperature strains:

$$\boldsymbol{b} = c \cdot \boldsymbol{a} + (T - T_0) \sum_{\alpha=0}^{m-1} c_\alpha (\boldsymbol{a} \cdot \boldsymbol{a}) \Big(E_6 + 2\frac{\boldsymbol{a}\boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\Big) \cdot \boldsymbol{i}_\alpha. \tag{11}$$

Setting all the coefficients c_α in (11) equal to zero, we arrive at the well-known linear relation between thermomechanical characteristics of the material $\boldsymbol{b} = c \cdot \boldsymbol{a}$ [1].

Inserting (9) into (8), we write the constitutive relations as

$$\boldsymbol{\sigma} = n(\boldsymbol{h}, \boldsymbol{h}^T) \cdot (\boldsymbol{h} - \boldsymbol{h}^T), \tag{12}$$

where $n(\boldsymbol{h}, \boldsymbol{h}^T) = c + \sum\limits_{\alpha=0}^{m-1} c_\alpha[\boldsymbol{i}_\alpha(\boldsymbol{h} + \boldsymbol{h}^T) + (\boldsymbol{h} + \boldsymbol{h}^T)\boldsymbol{i}_\alpha + (h_\alpha + h_\alpha^T)E_6]$.

If all the constants c_α are set equal to zero and strains are assumed to be infinitesimal ($\boldsymbol{h} = \boldsymbol{\varepsilon}$), Eq. (12) yields the well-known relations of linear thermoelasticity $\boldsymbol{\sigma} = c(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^T)$, used to formulate and solve boundary-value problems. In the constitutive relations (12), the tensor n depends not only on the current strains $\boldsymbol{h}$ but also on temperature. Hence, these relations take into account the temperature dependence of elastic properties of a medium.

We write the constitutive relations (12) in the physical three-dimensional space with allowance for one-to-one correspondence between six-dimensional vectors and second-rank tensors. As a result, we obtain the relation between the generalized tensor of true stresses and tensor of the Hencky strains

$$\Sigma = N^{\mathrm{IV}}(H, H^T) : (H - H^T), \tag{13}$$

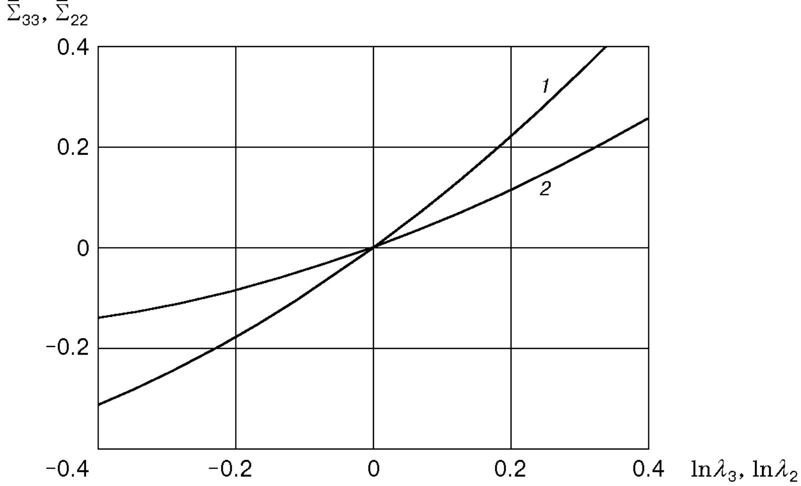

Fig. 1

where $N^{\rm IV} = C^{\rm IV} + \sum\limits_{\alpha=0}^{m-1} c_\alpha [I_\alpha(H + H^T) + (H + H^T)I_\alpha + (h_\alpha + h_\alpha^T)E^{\rm IV}]$, $C^{\rm IV}$ is the fourth-rank elastic tensor, I_α are the tensors of the canonical basis (1), H^T is the temperature-strain tensor, $E^{\rm IV} = \sum\limits_{\beta=0}^{5} I_\beta I_\beta$ is the fourth-rank unit tensor, and m is the dimension of the invariant subspace.

To construct relations (12) or (13) for a particular material, one needs experimental data on determining the elastic tensor from the initial portion of strain diagram ($\boldsymbol{h} \to \boldsymbol{0}$) [7]. To find the vector $\boldsymbol{h}^T$, it is necessary to perform experiments to determine thermal-expansion coefficients of materials, which are described in [8]. The constants c_α can be found from tests on uniaxial tension of materials along the axes of anisotropy. For an isotropic material, the constant c_0 can be determined from one test on tension in an arbitrary direction; for a transversely isotropic material, it is sufficient to perform two tests on tension in the direction of transverse isotropy and transverse direction. For an orthotropic material, the constants c_0, c_1, and c_2 can be found from three tests on tension along three principal axes of anisotropy. In the latter case, experimental diagrams are approximated by parabolas whose coefficients determine the desired constants.

For isothermal processes ($T = T_0$ and $H^T = 0$), the constitutive relations (13) become

$$\Sigma = \Big[C^{\rm IV} + \sum_{\alpha=0}^{m-1} c_\alpha (I_\alpha H + H I_\alpha + h_\alpha E^{\rm IV})\Big] : H. \tag{14}$$

By virtue of their nonlinearity, relations (14) govern the behavior of materials with different characteristics in tension and compression.

Let a cube from a transversely isotropic material be subjected to uniaxial strain along the axis of transverse isotropy $\boldsymbol{e}_3$. In this case, strains are characterized by the elongations $\lambda_3 \neq 1$ and $\lambda_1 = \lambda_2 = 1$ and the stresses Σ_{33} are given by

$$\Sigma_{33} = (C^{3333} + \sqrt{3}\,(c_0 + \sqrt{2}c_1)\ln\lambda_3)\ln\lambda_3,$$

where C^{3333} is the elastic modulus of the transversely isotropic material in the direction of the principal axis.

For deformation of the cube along the transverse axis ($\lambda_2 \neq 1$ and $\lambda_1 = \lambda_3 = 1$), the stresses Σ_{22} have the form

$$\Sigma_{22} = (C^{2222} + (\sqrt{3}/\sqrt{2})(\sqrt{2}c_0 - c_1)\ln\lambda_2)\ln\lambda_2.$$

Here C^{2222} is the elastic modulus of the transversely isotropic material in the transverse direction.

Figure 1 shows the dependences $\bar{\Sigma}_{33}(\ln\lambda_3)$ and $\bar{\Sigma}_{22}(\ln\lambda_2)$ (curves 1 and 2, respectively) for a material with the parameters $C^{2222}/C^{3333} = 0.5$, $c_0/C^{3333} = 0.25$, and $c_1/C^{3333} = 0.05$. One can see that the curves of tension and compression strains do not coincide. In contrast to the known models of materials with different compression and tension moduli [9], relations (14) describe a continuous change in the tangent modulus in the case of a varied deformation direction in the neighborhood of the point $\ln\lambda_3 = 0$, $\ln\lambda_2 = 0$.

We now consider triaxial isothermal deformation of a cube from a transversely isotropic material whose axes of anisotropy do not coincide with the direction of the cube edges. We denote the elongations of the edges by λ_1, λ_2, and λ_3. An analysis of relations (13) shows that, in this case of "pure" strain, the structure of the tensor $N^{\mathrm{IV}}(H)$ becomes different, which alters the type of initial anisotropy of the material. For an arbitrary combination of the elongations λ_1, λ_2, and λ_3, the material becomes triclinic, i.e, acquires the general type of property symmetry.

In summary, the constitutive relations of nonlinear anisotropic elasticity proposed take into account the nonlinear dependence of stresses on temperature, the dependence of the elastic properties of a material on temperature variation during deformation, the different behavior of an anisotropic material in tension and compression, and the change in the type of anisotropy during deformation.

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